JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 2, May 2014
http://dx.doi.org/10.14403/jcms.2014.27.2.227

ON GENERALIZED DERIVATIONS OF BE-ALGEBRAS

Kyung Ho Kim*

Abstract. In this paper, we introduce the notion of a generalized derivation in a BE-algebra, and consider the properties of generalized derivations. Also, we characterize the fixed set $Fix_d(X)$ and $Kerd$ by generalized derivations. Moreover, we prove that if d is a generalized derivation of a BE-algebra, every filter F is a d-invariant.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Q. P. Hu and X. Li introduced a wide class of abstracts: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [7], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a dualization of a generation of a BCK-algebras. In this paper, we introduce the notion of a generalized derivation in a BE- algebra, and consider the properties of generalized derivations. Also, we characterize the fixed set $Fix_d(X)$ and $Kerd$ by generalized derivations. Moreover, we prove that if d is a generalized derivation of BE-algebra, every filter F is a d-invariant.

2. Preliminaries

In what follows, let X denote an BE-algebra unless otherwise specified.

Received January 13, 2014; Accepted April 07, 2014.
2010 Mathematics Subject Classification: Primary 06F35, 03G25, 08A30.
Key words and phrases: BE-algebra, generalized derivation, self-distributive, filter, isotone.

By a *BE-algebra* we mean an algebra $(X;*,1)$ of type (2, 0) with a single binary operation "$*$" that satisfies the following identities: for any $x, y, z \in X$,

(BE1) $x * x = 1$ for all $x \in X$,
(BE2) $x * 1 = 1$ for all $x \in X$,
(BE3) $1 * x = x$ for all $x \in X$,
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra $(X, *, 1)$ is said to be *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A non-empty subset S of a BE-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. For any x, y in a BE-algebra X, we define $x \vee y = (y * x) * x$.

Let X be a BE-algebra. We define the binary operation "$\leq$" as the following,

$$x \leq y \Leftrightarrow x * y = 1$$

for all $x, y \in X$.

In a BE-algebra X, the following identities are true for all $x, y, z \in X$.

(p1) $x * (y * x) = 1$.
(p2) $x * ((x * y) * y)) = 1$.
(p3) Let X be a self-distributive BE-algebra. If $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$ for all $x, y, z \in X$.

Let X be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* of X if

(F1) $1 \in F$,
(F2) If $x \in F$ and $x * y \in F$, then $y \in F$.

Definition 2.1. A self-map d on X is called a *derivation* if

$$d(x * y) = (x * d(y)) \vee (d(x) * y)$$

for every $x, y \in X$.

Proposition 2.2. *Let d be a derivation of X. Then we have*

(1) $d(1) = 1$,
(2) $d(x) = d(x) \vee x$ *for all* $x \in X$.
(3) $x \leq d(x)$.

3. Generalized derivations of BE-algebras

DEFINITION 3.1. Let X be a BE-algebra. A map $D : X \to X$ is called a *generalized derivation* if there exists a derivation $d : X \to X$ such that

$$D(x * y) = (x * D(y)) \vee (d(x) * y)$$

for every $x, y \in X$.

EXAMPLE 3.2. Let $X = \{1, a, b\}$ be a set in which "$*$" is defined by

$*$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then X is a BE-algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1 \\ b & \text{if } x = a \\ a & \text{if } x = b. \end{cases}$$

Then it is easy to check that d is a derivation of a BE-algebra X. Also, define a map $D : X \to X$ by

$$D(x) = \begin{cases} 1 & \text{if } x = 1 \\ a & \text{if } x = a, b. \end{cases}$$

It is easy to verify that D is a generalized derivation of X.

EXAMPLE 3.3. Let $X = \{1, a, b, c\}$ be a set in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then X is a BE-algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, a \\ a & \text{if } x = b, c. \end{cases}$$

Then it is easy to check that d is a derivation of X. Also, define a map $D : X \to X$ by

$$D(x) = \begin{cases} 1 & \text{if } x = 1, a, c \\ a & \text{if } x = b. \end{cases}$$

It is easy to verify that D is a generalized derivation of X.

EXAMPLE 3.4. Let $X = \{1, a, b, c\}$ be a set in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	c	1	c
c	1	1	b	1

Then X is a BE-algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, b \\ c & \text{if } x = a, c. \end{cases}$$

Then it is easy to check that d is a derivation of X. Also, define a map $D : X \to X$ by

$$D(x) = \begin{cases} 1 & \text{if } x = 1, b \\ a & \text{if } x = a \\ c & \text{if } x = c. \end{cases}$$

Then it is easy to check that D is a generalized derivation of X.

EXAMPLE 3.5. Let $X = \{1, a, b, c\}$ be a set in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	1

Then X is a BE-algebra. Define a derivation $d : X \to X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, b, c \\ a & \text{if } x = a \end{cases}$$

and define a map $D : X \to X$ by

$$D(x) = \begin{cases} 1 & \text{if } x = 1, b \\ a & \text{if } x = a \\ c & \text{if } x = c \end{cases}$$

Then it is easy to check that D is a generalized derivation of X.

PROPOSITION 3.6. *Let d be a generalized derivation of X. Then we have*

(1) $D(1) = 1$,
(2) $D(x) = D(x) \vee x$ *for all* $x \in X$.

Proof. (1) Let D be a generalized derivation of X. Then we have

$$\begin{aligned} D(1) &= D(1 * 1) = (1 * D(1)) \vee (d(1) * 1) = D(1) \vee (1 * 1) \\ &= D(1) \vee 1 = (1 * D(1)) * D(1) = D(1) * D(1) = 1. \end{aligned}$$

(2) For all $x \in X$, we have

$$\begin{aligned} D(x) &= D(1 * x) = (1 * D(x)) \vee (d(1) * x) \\ &= D(x) \vee (1 * x) = D(x) \vee x. \end{aligned}$$

□

PROPOSITION 3.7. *Let D be a generalized derivation of X. Then the following identities hold:*

(1) $x \leq D(x)$ *for all* $x \in X$,
(2) *If X is a self-distributive BE-algebra, then* $D(x * y) = x * D(y)$ *for all* $x, y \in X$.

Proof. (1) By Proposition 3.6(2)and (BE4), we have for all $x \in X$,

$$\begin{aligned} x * D(x) &= x * (D(x) \vee x) = x * ((x * D(x)) * D(x)) \\ &= (x * D(x)) * (x * D(x)) = 1 \end{aligned}$$

which implies $x \leq D(x)$.

(2) By (1) and (p3), we have $x * y \leq x * D(y)$ and $d(x) * y \leq x * y$ by Proposition 2.2 (3). Hence we get

$$\begin{aligned} D(x * y) &= (x * D(y)) \vee (d(x) * y) \\ &= ((d(x) * y) * (x * D(y))) * (x * D(y)) \\ &= 1 * (x * D(y)) = x * D(y). \end{aligned}$$

□

PROPOSITION 3.8. *If D is a generalized derivation of X, then we have $D(D(x) * x) = 1$ for all $x \in X$.*

Proof. Let D be a generalized derivation of X. Then we have

$$\begin{aligned} D(D(x)*x) &= (D(x)*D(x)) \vee (d(D(x)))*x) \\ &= 1 \vee (d(D(x)))*x) = 1 \end{aligned}$$

for all $x \in X$. □

THEOREM 3.9. *Let D be a generalized derivation of X. Then D is one-to-one if and only if D is the identity map on X.*

Proof. Sufficiency is obvious. Suppose that D is one-to-one. For $x \in X$, we have

$$D(D(x)*x) = 1 = D(1)$$

and so $D(x)*x = 1$, i.e., $D(x) \leq x$. Since $x \leq D(x)$ for all $x \in X$, it follows that $D(x) = x$ so that D is the identity map. □

PROPOSITION 3.10. *Let X be a BE-algebra. A generalized derivation $D : X \to X$ is an identity map if it satisfies $x * D(y) = D(x) * y$ for all $x, y \in X$*

Proof. Let $x, y \in X$ be such that $x * D(y) = D(x) * y$. Now $D(x) = D(1*x) = 1*D(x) = D(1)*x = 1*x = x$. Thus d is an identity map. □

PROPOSITION 3.11. *Let X be a BE-algebra. Then*

$$x \leq D_n(D_{n-1}(...(D_2(D_1(x)))...))$$

for $n \in \mathbb{N}$, where $D_1, D_2, ..., D_n$ are generalized derivations of X.

Proof. For $n = 1$,

$$\begin{aligned} D_1(x) = D_1(1*x) &= (1*D_1(x)) \vee (d_1(1)*x) \\ &= D_1(x) \vee (1*x) = D_1(x) \vee x = (x*D_1(x))*D_1(x). \end{aligned}$$

Hence we have

$$x*D_1(x) = x*((x*D_1(x))*D_1(x)) = (x*D_1(x))*(x*D_1(x)) = 1$$

which implies $x*D_1(x) = 1$. Thus $x \leq D_1(x)$.

Let $n \in \mathbb{N}$ and $x \leq D_n(D_{n-1}(...(D_2(D_1(x)))...))$. For simplicity, let

$$T_n = D_n(D_{n-1}(...(D_2(D_1(x)))...)).$$

Then

$$\begin{aligned} D_{n+1}(T_n) = D_{n+1}(1*T_n) &= (1*D_{n+1}(T_n)) \vee (d_{n+1}(1)*T_n) \\ &= D_{n+1}(T_n) \vee T_n = (T_n*D_{n+1}(T_n))*D_{n+1}(T_n). \end{aligned}$$

Hence $T_n * T_{n+1} = 1$, which implies $T_n \leq T_{n+1}$. By assumption, $x \leq T_n \leq T_{n+1}$. □

Let D be a generalized derivation of X. Define a set $Fix_D(X)$ by

$$Fix_D(X) := \{x \in X \mid D(x) = x\}$$

for all $x \in X$.

PROPOSITION 3.12. *Let D be a generalized derivation of X. If $x \in Fix_D(X)$, then we have $(D \circ D)(x) = x$.*

Proof. Let $x \in Fix_D(X)$. Then we have

$$(D \circ D)(x) = D(D(x)) = D(x) = x.$$

This completes the proof. □

PROPOSITION 3.13. *Let D be a generalized derivation of a self-distributive BE-algebra X. If $y \in Fix_D(X)$, then we have $x * y \in Fix_D(X)$ for all $x \in X$.*

Proof. Let $y \in Fix_D(X)$. Then we have $D(y) = y$. Hence we have

$$\begin{aligned}
&D(x * y)\\
&= (x * D(y)) \vee (d(x) * y) = ((d(x) * y) * (x * y)) * (x * y)\\
&= (x * ((d(x) * y) * y)) * (x * y) = ((x * (d(x) * y)) * (x * y)) * (x * y)\\
&= ((x * d(x)) * (x * y)) * (x * y)) * (x * y) = ((1 * (x * y)) * (x * y)) * (x * y)\\
&= ((x * y) * (x * y)) * (x * y) = 1 * (x * y) = x * y.
\end{aligned}$$

This completes the proof. □

THEOREM 3.14. *Let X be a BE-algebra and let D_1, D_2 be two isotone generalized derivations on X. If $D(x) \in Fix_D(X)$, then $D_1 = D_2$ if and only if $Fix_{D_1}(X) = Fix_{D_2}(X)$.*

Proof. Let $D_1 = D_2$. Then $Fix_{D_1}(X) = Fix_{D_2}(X)$. Conversely, let $Fix_{D_1}(X) = Fix_{D_2}(X)$ and $D(x) \in Fix_D(X)$ for $x \in X$. Then $D_1(x) \in Fix_{D_1}(X) = Fix_{D_2}(X)$, and so $D_2(D_1(x)) = D_1(x)$. Also, $D_2(x) \in Fix_{D_2}(X) = Fix_{D_1}(X)$, and so $D_1(D_2(x)) = D_2(x)$. Since $x \leq D_1(x)$, we have $D_2(x) \leq D_2(D_1(x))$, and so $D_2(x) = D_1(D_2(x)) \leq D_2(D_1(x))$. Symmetrically, we have $D_2(D_1(x)) \leq D_1(D_2(x))$. Hence $D_1D_2 = D_2D_1$. It follows that $D_2(x) = D_1(D_2(x)) = D_2(D_1(x)) = D_1(x)$. □

Let D be a generalized derivation of X. Define a $KerD$ by

$$KerD = \{x \mid D(x) = 1\}$$

for all $x \in X$.

PROPOSITION 3.15. *Let D be a generalized derivation of X. Then $KerD$ is a subalgebra of X.*

Proof. Clearly, $1 \in KerD$, and so $KerD$ is nonempty. Let $x, y \in KerD$. Then $D(x) = 1$ and $D(y) = 1$. Hence we have

$$D(x*y) = (x*D(y)) \vee (d(x)*y) = (x*1) \vee (d(x)*y) = 1 \vee (d(x)*y) = 1,$$

and so $x * y \in KerD$. Thus $KerD$ is a subalgebra of X. □

A BE-algebra X is said to be *commutative* if for all $x, y \in X$,

$$(y * x) * x = (x * y) * y.$$

Proposition 3.16. *Let X be a commutative BE-algebra and let D be a generalized derivation. If $x \in KerD$ and $x \leq y$, then we have $y \in KerD$.*

Proof. Let $x \in KerD$ and $x \leq y$. Then $D(x) = 1$ and $x * y = 1$.

$$\begin{aligned} D(y) &= D(1 * y) = D((x * y) * y) \\ &= ((y * x) * D(x)) \vee (d(y * x) * x) \\ &= ((y * x) * 1) \vee (d(y * x) * x) \\ &= 1 \vee (d(y * x) * x) = 1, \end{aligned}$$

and so $y \in KerD$. This completes the proof. □

Theorem 3.17. *Let D be a generalized idempotent derivation of a self-distributive BE-algebra X. If D is isotone, then $KerD$ is a filter of X.*

Proof. Clearly, $1 \in KerD$. Let $x \in KerD$ and $x * y \in KerD$. Then we have $D(x) = D(x * y) = 1$, and so $1 = D(x * y) = x * D(y)$ by Proposition 3.7 (2). Hence $x \leq D(y)$. Since D is isotone, we get $1 = D(x) \leq D(D(y)) = D(y)$, which implies $D(y) = 1$. That is, $y \in KerD$. This completes the proof. □

Proposition 3.18. *Let D be a generalized derivation of X and $x, y \in KerD$. Then we have $x \vee y \in KerD$.*

Proof. Let D be a generalized derivation of X and $x, y \in KerD$. Then $D(x) = D(y) = 1$. Hence we have

$$\begin{aligned} D(x * y) &= (x * D(y)) \vee (D(x) * y) \\ &= (x * 1) \vee (1 * y) = 1 \vee y = 1. \end{aligned}$$

□

Proposition 3.19. *Let D be a generalized derivation of X and $y \in KerD$. Then we have $x * y \in KerD$ for all $x \in X$.*

Proof. Let D be a generalized derivation of X and $y \in KerD$. Then $D(y) = 1$. Hence we have for all $x \in X$,

$$\begin{aligned} D(x * y) &= (x * D(y)) \vee (d(x) * y) \\ &= (x * 1) \vee (d(x) * y) = 1 \vee d(x) * y) = 1. \end{aligned}$$

□

Proposition 3.20. *Let D be a generalized derivation of X. If D is one-to-one, then $KerD = 1$.*

Proof. Suppose that D is one-to-one and $x \in Ker(D)$. Then $D(x) = 1 = D(1)$, and thus $x = 1$, i.e., $KerD = \{1\}$. □

Definition 3.21. Let X be a BE-algebra. A self-map D is *isotone* if $x \leq y$ implies $D(x) \leq D(y)$.

Proposition 3.22. *Let D be a generalized derivation of X. If D is an endomorphism of X, then D is isotone.*

Proof. Let $x \leq y$. Then $x * y = 1$. Hence we have

$$D(x) * D(y) = D(x * y) = D(1) = 1,$$

which implies $D(x) \leq D(y)$. This completes the proof. □

Proposition 3.23. *Let D be an isotone generalized derivation of X. If $x \leq y$ and $x \in KerD$, then $y \in KerD$.*

Proof. Let $x \leq y$ and $x \in KerD$. Then we have $D(x) = 1$, and so

$$1 = D(x) \leq D(y),$$

which implies $D(y) = 1$. □

Definition 3.24. Let X be a BE-algebra. A nonempty subset F of X is said to be a *D-invariant* if $D(F) \subseteq F$ where $D(F) = \{D(x) \mid x \in X\}$.

Proposition 3.25. *Let X be a BE-algebra and let D be a generalized derivation of X. Then every filter F is a D-invariant.*

Proof. Let F be a filter of X. Let $y \in D(F)$. Then $y = D(x)$ for some $x \in F$. It follows from Proposition 3.7(1) that $x * y = x * D(x) = 1 \in F$, which implies $y \in F$. Thus $D(F) \subseteq F$. Hence F is D-invariant. □

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Department of Mathematics
Korea National University of Transportation
Chungju 380-702, Republic of Korea
E-mail: ghkim@ut.ac.kr